

Effects of self-phase modulation on weak nonlinear optical quantum gates

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A possible two-qubit gate for optical quantum computing is the parity gate based on the weak Kerr effect. Two photonic qubits modulate the phase of a coherent state, and a quadrature measurement of the coherent state reveals the parity of the two qubits without destroying the photons. This can be used to create so-called cluster states, a universal resource for quantum computing. Here, the effect of self-phase modulation on the parity gate is studied, introducing generating functions for the Wigner function of a modulated coherent state. For materials with non-EIT-based Kerr nonlinearities, there is typically a self-phase modulation that is half the magnitude of the cross-phase modulation. Therefore, this effect cannot be ignored. It is shown that for a large class of physical implementations of the phase modulation, the quadrature measurement cannot distinguish between odd and even parity. Consequently, weak nonlinear parity gates must be implemented with physical systems where the self-phase modulation is negligible.

I. INTRODUCTION

Linear optical quantum computing with photonic qubits has generated considerable interest in recent years [1, 2]. However, it has become clear that from a scaling perspective, some optical nonlinearity is extremely desirable. This can be achieved in a number of ways, either by using the coupling of photons with matter qubits [3, 4, 5], quantum Zeno gates [6] or by using optical nonlinearities [7, 8, 9]. To date, analysis of weak nonlinear optical gates has not taken into account self-phase modulation, mainly because the envisaged implementation is materials with an electromagnetically induced transparency (EIT). It is known that in such systems a large cross-phase modulation (Kerr nonlinearity) can be achieved without any self-phase modulation [10]. However, in general nonlinear optical media self-phase modulation effects are present, and are typically of the order of half the cross-phase modulation. This is known as weak-wave retardation [11, 12].

In this paper, we analyse the effects of self-phase modulation on the operation of the weak nonlinear optical parity gate, which was first introduced by Barrett et al. [7]. To this end, the classical and quantum theory of materials with third-order optical nonlinearities is reviewed, and we develop a description of the self-phase modulation of a coherent state in terms of its Wigner function and the corresponding marginal probability distributions for the quadratures. We then give a modified, more realistic description of the parity gate, and show that a typical amount of self-phase modulation destroys the distinguishability between even and odd parity. We conclude that the nonlinearities in weak nonlinear optical parity gates must have negligible self-phase modulation.

The original optical parity gate based on conditional phase shifts without self-phase modulation effects can be summarized as follows [7]: Two single-photon qubits each couple to an optical mode in a coherent state $|\alpha\rangle$, such

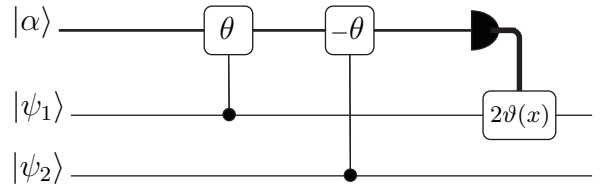


FIG. 1: Controlled phase shifts θ and $-\theta$ on a coherent state $|\alpha\rangle$, followed by an x -quadrature measurement induce a deterministic and nondestructive parity measurement. A corrective phase shift of $2\vartheta(x)$ must be applied when the measurement outcome x indicates odd parity. The conditional phase shifts are generated by a cross-Kerr nonlinearity.

that a phase shift is induced in the coherent state whenever the qubit is in the logical state $|1\rangle$. Moreover, the first qubit induces a controlled phase shift θ , while the second qubit induces a controlled phase shift $-\theta$ (see Fig. 1). The input state $(c_{00}|00\rangle + c_{01}|01\rangle + c_{10}|10\rangle + c_{11}|11\rangle)|\alpha\rangle$ transforms into the output state

$$c_{00}|00\rangle|\alpha\rangle + c_{01}|01\rangle|\alpha e^{-i\theta}\rangle + c_{10}|10\rangle|\alpha e^{i\theta}\rangle + c_{11}|11\rangle|\alpha\rangle.$$

If we assume that $\alpha \in \mathbb{R}$, then an \hat{x} quadrature measurement of the coherent state will project the qubit states either onto the even parity subspace $(c_{00}|00\rangle + c_{11}|11\rangle)$, or onto the odd parity subspace $(c_{01}|01\rangle + e^{2i\vartheta(x)}c_{10}|10\rangle)$. When the measurement outcome x indicates projection onto the odd subspace, a known corrective phase shift $2\vartheta(x)$ must be applied. Hence this is a deterministic parity gate that can be used to build cluster states [3, 13], which are a universal resource for quantum computing. The requirements for this gate to work with sufficiently high fidelity is that $|\langle\alpha|\alpha e^{i\theta}\rangle| \ll 1$, or $\alpha\theta^2 \gtrsim 1$. In the modified protocol of Spiller et al., [9] the requirement becomes $\alpha\theta > 1$. This scaling is important in practical implementations, where we require α to be reasonably small [14]. The coherent state is often called the *bus*, and the conditional phase shifts are implemented with optically active materials such as Kerr nonlinearities.

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II. THEORY OF KERR NONLINEARITIES

The weak nonlinear optical parity gate for photonic qubits uses the optical Kerr effect. In this section we review the classical and quantum theory of the third-order nonlinear interaction that leads to the Kerr effect, and we present a convenient way to describe the Kerr effect on coherent states using Wigner functions.

A. Classical and quantum theory

A detailed review of the classical theory of the Kerr effect can be found in Boyd [12], and here we give a brief description of the relevant physics. For a lossless and dispersionless nonlinear medium, the optical response can be expressed as a power series in the electric field of the polarization field in the medium:

$$P(t) = \chi^{(1)}E(t) + \chi^{(2)}E(t)^2 + \chi^{(3)}E(t)^3. \quad (\text{II.1})$$

where $P(t) = \sum_k P(\omega_k)e^{-i\omega_k t} + \text{c.c.}$ and $E(t) = \sum_n E(\omega_n)e^{-i\omega_n t} + \text{c.c.}$ are multi-mode expansions of the polarization and electric field. Here, we are particularly interested in the third order term $P^{(3)}(t) = \chi^{(3)}E(t)^3$. The polarizability and the electric fields are of course vector fields with three spatial components, and in general we should write

$$P_i^{(3)}(t) = \sum_{jkl} \chi_{ijkl}^{(3)} E_j(t) E_k(t) E_l(t). \quad (\text{II.2})$$

The nonlinear susceptibility $\chi_{ijkl}^{(3)}$ is then a fourth-rank tensor with 81 independent components. However, most materials are highly symmetric, and the actual number of independent components is far less. Here, we take $\chi^{(3)}$ to be independent of the orientation of the fields, and treat the electric field and the polarization as scalar quantities.

The nonlinear medium required for the optical parity gate couples two modes, a and b , with possibly different frequencies ω_a and ω_b . The frequencies may be identical, as long as the modes are distinguishable (e.g., they have different spatial directions). When we substitute this two-mode expansion of the electric field into Eq. (II.1) and collect all terms that are proportional to $e^{-i\omega_a t}$, the third order of the polarization field consists of two terms, namely the *cross-phase* modulation (XPM)

$$P_{XPM}^{(3)}(\omega_a) = 6\chi^{(3)}E(\omega_a)|E(\omega_b)|^2, \quad (\text{II.3})$$

and the *self-phase* modulation (SPM)

$$P_{SPM}^{(3)}(\omega_a) = 3\chi^{(3)}E(\omega_a)|E(\omega_a)|^2. \quad (\text{II.4})$$

Here, we have suppressed the time dependence. In XPM, the polarization of mode a depends on the intensity of the field in mode b , while in SPM the polarization in mode a depends on the intensity in mode a . Notice the relative factor of two in the magnitude of the phase modulations.

This means that for general isotropic nonlinear media the self-phase modulation is typically half the size of the cross-phase modulation.

The index of refraction n for a specific mode in a medium can be defined in terms of an effective susceptibility χ_{eff} such that $n^2 = 1 + 4\pi\chi_{\text{eff}}$. Taking into account the third-order polarizability P (and assuming a vanishing second-order susceptibility) we have for mode a :

$$\chi_{\text{eff}} \equiv \chi^{(1)} + 3\chi^{(3)}|E(\omega_a)|^2 + 6\chi^{(3)}|E(\omega_a)|^2, \quad (\text{II.5})$$

In other words, there will be a phase shift in mode a that is proportional not only to the regular phase shift ($\chi^{(1)}$), but also proportional to the intensity $|E|^2$ in mode a and in mode b .

Quantum mechanically, phase shifts are generated by the number operator of a given optical mode:

$$e^{i\varphi\hat{a}^\dagger\hat{a}}\hat{a}e^{-i\varphi\hat{a}^\dagger\hat{a}} = \hat{a}e^{-i\varphi}. \quad (\text{II.6})$$

When the phase shift of mode a is proportional to the intensity of the field in mode b , one would expect the phase shift generator $\hat{n}_a = \hat{a}^\dagger\hat{a}$ to be multiplied by the number operator of mode b . Indeed, the XPM and SPM effects are described by the interaction Hamiltonians

$$H_{XPM} = \theta\hat{a}^\dagger\hat{a}\hat{b}^\dagger\hat{b} = \theta\hat{n}_a\hat{n}_b, \quad (\text{II.7})$$

and

$$H_{SPM} = \phi_a(\hat{a}^\dagger\hat{a})^2 + \phi_b(\hat{b}^\dagger\hat{b})^2 = \phi_a\hat{n}_a^2 + \phi_b\hat{n}_b^2. \quad (\text{II.8})$$

The coupling constants θ and ϕ_j ($j = a, b$) are proportional to $6\chi^{(3)}$ and $3\chi^{(3)}$, respectively. In the remainder of this paper we set $\phi_a = \phi_b = \phi$, and towards the end we shall set $\theta = 2\phi$. We note a couple of things: First, the signs of θ and ϕ must be identical, since they are both proportional to $\chi^{(3)}$. This will become important when we analyze the weak nonlinear parity gate. And secondly, since $[H_{XPM}, H_{SPM}] = 0$, we can treat the self-phase modulation independent from the cross-phase modulation.

Given the interaction Hamiltonian H_{SPM} , we can calculate the effect of SPM on a coherent state:

$$e^{-i\phi\hat{n}^2}|\alpha\rangle = e^{-|\alpha|^2/2} \sum_{n=0}^{\infty} \frac{\alpha^n e^{-i\phi n^2}}{\sqrt{n!}} |n\rangle. \quad (\text{II.9})$$

However, the oft-needed inner products $\langle x_\lambda | e^{-i\phi\hat{n}^2} | \alpha \rangle$ and $\langle \beta | e^{-i\phi\hat{n}^2} | \alpha \rangle$ cannot be evaluated analytically. To find the effect of SPM on measured quantities, we should instead transform the corresponding observables using $\exp(-iH_{SPM})$.

The operator transformations for the creation and annihilation operators of the field modes, given the XPM and SPM interaction Hamiltonians (denoted by φA), can formally be written as

$$e^{i\varphi A}\hat{a}e^{-i\varphi A} = \hat{a} + i\varphi[A, \hat{a}] + \frac{(i\varphi)^2}{2!}[A, [A, \hat{a}]] + \dots \quad (\text{II.10})$$

It is straightforward to show that

$$\begin{aligned} e^{iH_{XPM}} \hat{a} e^{-iH_{XPM}} &= \hat{a} e^{-i\theta \hat{n}_b} \\ e^{iH_{SPM}} \hat{a} e^{-iH_{SPM}} &= e^{-i\theta(2\hat{n}_a+1)} \hat{a} \\ e^{iH_{SPM}} \hat{a}^\dagger e^{-iH_{SPM}} &= \hat{a}^\dagger e^{i\theta(2\hat{n}_a+1)}. \end{aligned} \quad (\text{II.11})$$

For the SPM case, the operators are ordered such that all creation operators are on the left, the phase operators $\exp[-i\phi \hat{n}_a]$ are in the center, and the annihilation operators are on the right. This will be convenient later, when we evaluate expectation values with respect to coherent states.

The quadrature operators of the electromagnetic field can be defined as [15]

$$\hat{x}_\lambda = \frac{1}{\sqrt{2}} (\hat{a} e^{-i\lambda} + \hat{a}^\dagger e^{i\lambda}). \quad (\text{II.12})$$

We can construct the canonical momentum to this operator as $\hat{x}_{\lambda+\pi/2}$, since $[\hat{x}_\lambda, \hat{x}_{\lambda+\pi/2}] = i$. The effect of SPM on the quadrature operator results in a transformed operator \hat{x}'_λ

$$\hat{x}'_\lambda = \frac{1}{\sqrt{2}} \left(e^{-i\lambda-i\phi(2\hat{n}+1)} \hat{a} + \hat{a}^\dagger e^{i\lambda+i\phi(2\hat{n}+1)} \right). \quad (\text{II.13})$$

The expression in Eq. (II.13) allows us to evaluate the moments of \hat{x}'_λ , and consequently we can calculate the mean, variance, skewness, and kurtosis of the probability distribution of measurement outcomes for \hat{x}'_λ .

B. Wigner function of self-phase modulated coherent state

In the weak nonlinear parity gate, the SPM affects the coherent state that acts as the bus mode. The qubits are single photons, and are affected only in a trivial way by the SPM, in that it induces a known phase shift in the qubit states. In this section, we derive the Wigner function for a self-phase modulated coherent state.

Any single-mode state of the electromagnetic field can be written in a photon number expansion $|\Psi\rangle = \sum_n A_n |n\rangle$. Furthermore, we can insert a resolution of the identity in terms of coherent states such that

$$|\Psi\rangle = \sum_{n=0}^{\infty} A_n |n\rangle = \sum_{n=0}^{\infty} \int \frac{d^2\beta}{\pi} A_n |\beta\rangle \langle\beta|n\rangle. \quad (\text{II.14})$$

For $|\Psi\rangle$ a coherent state experiencing SPM, we found that

$$A_n = e^{-|\alpha|^2/2} \frac{\alpha^n}{\sqrt{n!}} e^{-i\phi n^2}. \quad (\text{II.15})$$

Using the inner product $\langle\beta|n\rangle = e^{-|\beta|^2/2} \beta^{*n} / \sqrt{n!}$ we find

$$|\Psi\rangle = \sum_{n=0}^{\infty} \int \frac{d^2\beta}{\pi} e^{-(|\alpha|^2+|\beta|^2)/2} \frac{(\alpha\beta^*)^n}{n!} e^{-i\phi n^2} |\beta\rangle. \quad (\text{II.16})$$

The factor $e^{-i\phi n^2}$ can be written as a power series $\sum_k (-i\phi)^k n^{2k} / k!$:

$$|\Psi\rangle = \sum_{k,n=0}^{\infty} \int \frac{d^2\beta}{\pi} e^{-(|\alpha|^2+|\beta|^2)/2} \frac{(\alpha\beta^*)^n}{n!} \frac{(-i\phi)^k n^{2k}}{k!} |\beta\rangle. \quad (\text{II.17})$$

This expansion allows us to remove the factor n^{2k} by applying the differential operator $(\alpha\partial_\alpha)^{2k}$:

$$|\Psi\rangle = e^{-|\alpha|^2/2} \sum_{k=0}^{\infty} \frac{(-i\phi)^k (\alpha\partial_\alpha)^{2k}}{k!} \int \frac{d^2\beta}{\pi} e^{-|\beta|^2/2+\alpha\beta^*} |\beta\rangle. \quad (\text{II.18})$$

Rewriting the power series in k as an exponential, we have

$$|\Psi\rangle = e^{-|\alpha|^2/2} U_\alpha(\phi) \int \frac{d^2\beta}{\pi} e^{-|\beta|^2/2+\alpha\beta^*} |\beta\rangle, \quad (\text{II.19})$$

where

$$U_\alpha(\phi) = \exp \left[-i\phi (\alpha\partial_\alpha)^2 \right]. \quad (\text{II.20})$$

The Wigner function of this state is constructed according to ($\hbar = 1$)

$$W(q, p) = \frac{1}{\pi} \int_{-\infty}^{\infty} dx e^{-2ipx} \langle q-x|\Psi\rangle \langle\Psi|q+x\rangle, \quad (\text{II.21})$$

and the marginal probability distribution of the q quadrature is given by

$$P(q) = \int_{-\infty}^{\infty} W(q, p) dp. \quad (\text{II.22})$$

To calculate the Wigner function, we first evaluate $\langle x|\Psi\rangle$, using $\langle x|\alpha\rangle = \pi^{-1/4} \exp[-\frac{1}{2}(x-\sqrt{2}\alpha)^2 + \frac{1}{2}\alpha(\alpha-\alpha^*)]$ [15]:

$$\begin{aligned} \langle x|\Psi\rangle &= e^{-\frac{1}{2}|\alpha|^2} U_\alpha(\phi) \int \frac{d^2\beta}{\pi \sqrt[4]{\pi}} e^{-\frac{1}{2}x^2 + \sqrt{2}\beta x + \alpha\beta^* - |\beta|^2 - \frac{1}{2}\beta^2} \\ &= e^{-\frac{1}{2}|\alpha|^2} U_\alpha(\phi) \frac{e^{-\frac{1}{2}x^2 + \sqrt{2}\alpha x - \frac{1}{2}\alpha^2}}{\sqrt[4]{\pi}} \\ &\equiv e^{-\frac{1}{2}|\alpha|^2} U_\alpha(\phi) G_\alpha(x), \end{aligned} \quad (\text{II.23})$$

where we defined $G_\alpha(x)$ as a generating function for the Wigner function such that

$$\begin{aligned} W(q, p) &= \frac{e^{-|\alpha|^2}}{\pi} U_\beta(\phi) U_\gamma^\dagger(\phi) \\ &\times \int dx e^{-2ipx} G_\beta(q-x) G_\gamma^*(q+x) \Big|_{\beta, \gamma=\alpha}. \end{aligned} \quad (\text{II.24})$$

The integral over x can be evaluated to yield

$$W(q, p) = \frac{e^{-|\alpha|^2}}{\pi} U_\beta(\phi) U_\gamma^\dagger(\phi) K_{\beta, \gamma}(q, p) \Big|_{\beta, \gamma=\alpha}, \quad (\text{II.25})$$

with the generating function

$$K_{\beta, \gamma}(q, p) = e^{-p^2 - q^2 + i\sqrt{2}(\beta-\gamma^*)p + \sqrt{2}(\beta+\gamma^*)q - \beta\gamma^*}. \quad (\text{II.26})$$

The marginal probability distribution over the q quadrature for the SPM coherent state then becomes

$$P(q) = \frac{e^{-|\alpha|^2}}{\sqrt{\pi}} U_\beta(\phi) U_\gamma^\dagger(\phi) L_{\beta,\gamma}(q) \Big|_{\beta,\gamma=\alpha}, \quad (\text{II.27})$$

with the generating function

$$L_{\beta,\gamma}(q) = e^{-q^2 + \sqrt{2}(\beta + \gamma^*)q - \frac{1}{2}(\beta - \gamma^*)^2 - \beta\gamma^*}. \quad (\text{II.28})$$

The Wigner function and the marginal probability distribution can be found by applying the differential operators U_α to the respective generating functions. In the case of the weak nonlinear parity gate, we are interested in the case where $\alpha\theta^2 \simeq 1$, and a first-order expansion of U_α will not be sufficient to properly evaluate the marginal probability distribution. Instead, it has to be evaluated numerically.

III. THE PARITY GATE WITH SELF-PHASE MODULATION

The original weak nonlinear parity gate as described above is invariant under SPM. This is because the coupling constant for SPM and XPM have the same sign, and in the two successive stages of the protocol two conditional phase shifts θ with opposite signs are used. Unfortunately, it is generally not possible to change the sign of the conditional phase shift. The nonlinear susceptibility $\chi^{(3)}$ is a material constant that cannot readily be changed. Using different materials with opposite nonlinearities seems highly impractical. In the case of EIT, the phase shift is proportional to the detuning of the qubit mode with the relevant transition in the EIT medium. Since both the frequency of the single photon and the transition frequency are fixed, it is not possible to switch the sign of the nonlinearity in the weak nonlinear parity gate.

However, it is possible to construct the parity gate with two identical conditional phase shifts. Let the input state of the two qubits and the coherent bus again be given by $(c_{00}|00\rangle + c_{01}|01\rangle + c_{10}|10\rangle + c_{11}|11\rangle)|\alpha\rangle$. The two coherent phase shifts now generate the state

$$c_{00}|00\rangle|\alpha\rangle + (c_{01}|01\rangle + c_{10}|10\rangle)|\alpha e^{i\theta}\rangle + c_{11}|11\rangle|\alpha e^{2i\theta}\rangle.$$

If $\alpha \in \mathbb{R}$, instead of measuring the x quadrature, we measure \hat{x}_λ , where we now have to choose λ such that

$$\langle\alpha|\hat{x}_\lambda|\alpha\rangle = \langle\alpha e^{2i\theta}|\hat{x}_\lambda|\alpha e^{2i\theta}\rangle. \quad (\text{III.1})$$

This leads to the requirement $\cos\lambda = \cos(2\theta - \lambda)$, which is satisfied for $\lambda = \theta$. Constructed this way, the parity gate is rotated by an angle θ in phase space, and the roles of even and odd parity are reversed, in that the corrective phase shift is now applied to the even parity outcome. The variance is not affected by the rotation, and the distinguishability requirement is the same as in the original setup.

When the weak nonlinear parity gate is constructed with two identical conditional phase shifts, the SPM no longer cancels, and instead of evaluating the expectation values of \hat{x}_λ , we need to work with \hat{x}'_λ defined in Eq. (II.13). The expectation values of \hat{x}'_λ with respect to $|\alpha\rangle$ and $|\alpha e^{2i\theta}\rangle$ will be different from the expectation values of \hat{x}_λ , and we have to recalculate the value of λ for which the even parity contributions overlap. For a coherent state with $\alpha = r e^{i\xi}$ the mean is

$$\langle\alpha|\hat{x}'_\lambda|\alpha\rangle = \sqrt{2}r e^{-r^2(1-\cos 4\phi)} \cos(r^2 \sin 4\phi + 2\phi - \xi + \lambda). \quad (\text{III.2})$$

Next, we calculate

$$\langle\alpha|\hat{x}'_\lambda|\alpha\rangle = \langle\alpha e^{2i\theta}|\hat{x}'_\lambda|\alpha e^{2i\theta}\rangle \quad (\text{III.3})$$

for $\alpha = r \in \mathbb{R}$. This leads to $\cos(r^2 \sin 4\phi + 2\phi - 2\theta + \lambda) = \cos(r^2 \sin 4\phi + 2\phi + \lambda)$, or

$$\lambda = \theta - r^2 \sin 4\phi - 2\phi. \quad (\text{III.4})$$

Note that we have evaluated $\langle\hat{x}'_\lambda\rangle$ with $\phi \rightarrow 2\phi$ since the SPM effect occurs *twice* in the parity gate. We see that the value of λ changes dramatically, since it now depends on the magnitude r of the coherent state in the bus mode (λ will generally be a large multiple of 2π). This means that λ must be set with extremely high precision. For the sake of the argument, we assume here that λ can be set exactly as in Eq. (III.4).

The next step in the analysis of the weak nonlinear parity gate with SPM is to evaluate the variance with respect to the three different coherent bus states. To this end, we calculate $\langle(\hat{x}'_\lambda)^2\rangle$, again with $\phi \rightarrow 2\phi$. The expectation value of the operator $(\hat{x}'_\lambda)^2$ is

$$\langle(\hat{x}'_\lambda)^2\rangle = \frac{1}{2} + r^2 + r^2 e^{-r^2(1-\cos 8\phi)} \cos(2\lambda - 2\xi + 8\phi + r^2 \sin 8\phi). \quad (\text{III.5})$$

We use the value of λ given in Eq. (III.4) and $\theta = 2\phi$, and we make the approximations $\cos x = 1 - x^2/2$ and

$\sin x = x - x^3/6$. The variance of \hat{x}'_λ for the three coherent states $|\alpha\rangle$, $|\alpha e^{i\theta}\rangle$, and $|\alpha e^{2i\theta}\rangle$ then becomes

$$(\Delta \hat{x}'_\lambda)_0^2 = \frac{1}{2} + r^2 + r^2 e^{-8r^2\theta^2} \cos(4\theta - 8r^2\theta^3) - 2r^2 e^{-4r^2\theta^2} \cos^2 \theta, \quad (\text{III.6})$$

$$(\Delta \hat{x}'_\lambda)_\theta^2 = \frac{1}{2} + r^2 + r^2 e^{-8r^2\theta^2} \cos(2\theta - 8r^2\theta^3) - 2r^2 e^{-4r^2\theta^2}, \quad (\text{III.7})$$

$$(\Delta \hat{x}'_\lambda)_{2\theta}^2 = \frac{1}{2} + r^2 + r^2 e^{-8r^2\theta^2} \cos(8r^2\theta^3) - 2r^2 e^{-4r^2\theta^2} \cos^2 \theta, \quad (\text{III.8})$$

It is clear that the variances for the two different even parity states are not identical, and the measurement will generally cause an outcome-dependent rotation in the even parity subspace. However, since the measurement outcome is known (it is needed to determine the corrective phase shift), the state of the qubits remains pure. The gate introduces a so-called *tilting* error, which can be absorbed in an adaptive strategy for cluster state generation [16]. The corrective phase shift needed for the gate operation can be calculated using Eq. (II.23) (see also Rohde et al. [17]).

IV. RESOLUTION CRITERIUM

Given the setup discussed in the previous section, we now ask what is the parameter range that provides good distinguishability between even and odd parity projections. Distinguishability of the parities requires that the means of the two distributions (of even and odd parity) must be larger than the sum of the variances of the parity distributions:

$$|\langle \hat{x}'_\lambda \rangle_e - \langle \hat{x}'_\lambda \rangle_o| > (\Delta \hat{x}'_\lambda)_o + \max(\Delta \hat{x}'_\lambda)_e. \quad (\text{IV.1})$$

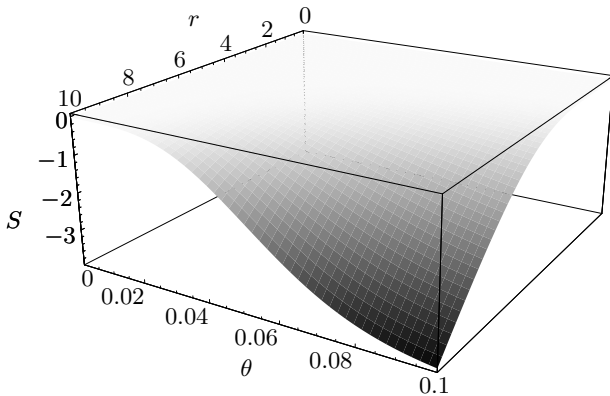


FIG. 2: The difference S between the separation of the means $|\langle \hat{x}'_\lambda \rangle_e - \langle \hat{x}'_\lambda \rangle_o|$ and the sum of the variances $(\Delta \hat{x}'_\lambda)_e + (\Delta \hat{x}'_\lambda)_o$ in arbitrary units. This quantity is strictly negative, which means that a quadrature measurement can never distinguish between even and odd parity when the self-phase modulation is half the size of the cross-phase modulation.

where the subscripts e and o denote evaluation with respect to the even or odd parity distribution, respectively, and $\max(\Delta \hat{x}'_\lambda)_e$ indicates that we need to evaluate the inequality using the largest variance from the pair $(\Delta \hat{x}'_\lambda)_0^2$ and $(\Delta \hat{x}'_\lambda)_{2\theta}^2$. This is the resolution criterium. Using the means and variances of the previous section (where $\theta = 2\phi$), we find that the requirement in Eq. (IV.1) is never satisfied. Since

$$(\Delta \hat{x}'_\lambda)_o + \max(\Delta \hat{x}'_\lambda)_e \geq 2 \min(\Delta \hat{x}'_\lambda)_j \quad (\text{IV.2})$$

with $j \in \{0, \theta, 2\theta\}$, we can instead evaluate

$$|\langle \hat{x}'_\lambda \rangle_e - \langle \hat{x}'_\lambda \rangle_o|^2 > 4 \min(\Delta \hat{x}'_\lambda)_j^2 \quad (\text{IV.3})$$

to show that the two probability distributions are never resolved in a measurement of the \hat{x}_λ quadrature. We plot the difference S between the mean separation and this variance in Fig. 2 (for r and θ in arbitrary units), and see that this function never exceeds zero. This means that for the weak nonlinear parity gate the SPM must be negligible.

The SPM is a non-Gaussian operation, and *a priori* there is no reason for the two probability distributions to obey the resolution criterium in Eq. (IV.1). The SPM may change the distributions in such a way that the variance becomes large, but the distributions still give two distinct peaks. Such a distribution is said to be *leptokurtic*, with a positive kurtosis. On the other hand, in a *platykurtic* distribution, where the kurtosis is negative, the values tend to lie further away from the mean. In this case the resolution criterium based on the variance is valid. We also need to consider the skewness, or asymmetry, of the distributions. For highly asymmetric distributions the variance will not be a good resolution criterium. In order to convince ourselves that the criterium is valid, we calculate the skewness and the kurtosis of the probability distributions.

The skewness γ_1 of a probability distribution can be expressed in terms of the third moment about the mean μ_3 and the standard deviation $\sigma = \Delta \hat{x}'_\lambda$:

$$\gamma_1 = \frac{\mu_3}{(\Delta \hat{x}'_\lambda)^3} \quad \text{and} \quad \mu_3 = \langle (\hat{x}'_\lambda - \langle \hat{x}'_\lambda \rangle)^3 \rangle. \quad (\text{IV.4})$$

A positive skewness implied an elongated tail towards the positive end of the parameter space. The third moment about the mean can be written in terms of expectation values of the higher order moments of \hat{x}'_λ :

$$\mu_3 = \langle (\hat{x}'_\lambda)^3 \rangle - 3\langle (\hat{x}'_\lambda)^2 \rangle \langle \hat{x}'_\lambda \rangle + 2\langle \hat{x}'_\lambda \rangle^3. \quad (\text{IV.5})$$

Similarly, the kurtosis γ_2 of a probability distribution can be defined in terms of the fourth moment about the mean μ_4 and the standard deviation $\sigma = \Delta\hat{x}'_\lambda$:

$$\gamma_2 = \frac{\mu_4}{(\Delta\hat{x}'_\lambda)^4} - 3 \quad \text{and} \quad \mu_4 = \langle (\hat{x}'_\lambda - \langle \hat{x}'_\lambda \rangle)^4 \rangle. \quad (\text{IV.6})$$

In terms of the higher order moments of \hat{x}'_λ , we can write

μ_4 as

$$\mu_4 = \langle (\hat{x}'_\lambda)^4 \rangle - 4\langle (\hat{x}'_\lambda)^3 \rangle \langle \hat{x}'_\lambda \rangle + 6\langle (\hat{x}'_\lambda)^2 \rangle \langle \hat{x}'_\lambda \rangle^2 - 3\langle \hat{x}'_\lambda \rangle^4.$$

We evaluate the expectation values $\langle (\hat{x}'_\lambda)^3 \rangle$ and $\langle (\hat{x}'_\lambda)^4 \rangle$ with respect to coherent states $|\alpha\rangle$, where $\alpha = r e^{i\xi}$:

$$\begin{aligned} \langle (\hat{x}'_\lambda)^3 \rangle &= \frac{r^3}{\sqrt{2}} e^{-r^2(1-\cos 12\phi)} \cos(3\lambda - 3\xi + 18\phi + r^2 \sin 12\phi) + \frac{3(r^3 + r)}{\sqrt{2}} e^{-r^2(1-\cos 4\phi)} \cos(\lambda - \xi + 6\phi + r^2 \sin 4\phi) \\ \langle (\hat{x}'_\lambda)^4 \rangle &= \frac{r^4}{2} e^{-r^2(1-\cos 16\phi)} \cos(4\lambda - 4\xi + 32\phi + r^2 \sin 16\phi) \\ &\quad + (2r^4 + 3r^2) e^{-r^2(1-\cos 8\phi)} \cos(2\lambda - 2\xi + 16\phi + r^2 \sin 8\phi) + \frac{3}{2}r^4 + 3r^2 + \frac{3}{4}. \end{aligned} \quad (\text{IV.7})$$

These expressions are exact. Using the value $\lambda = \theta - 2\phi - r^2 \sin 4\phi$ and $\theta = 2\phi$, we plot the skewness and kurtosis as a function of r and θ in Figs. 3 and 4. In the parameter regime $r\theta^2$ where the non-phase modulated parity gate operates, the skewness is practically zero and the kurtosis is negative. This indicates that the effect of SPM indeed destroys the distinguishability between the probability distributions for even and odd parity.

V. SQUEEZING

Finally, we may salvage the weak nonlinear parity gate by employing quadrature squeezing of the bus mode after the effects of SPM. The squeezing must be in the \hat{x}_λ direction such that the peaks in the marginal probability distribution P become narrower. Typically, when ζ is the squeezing parameter, the variance is reduced by a factor

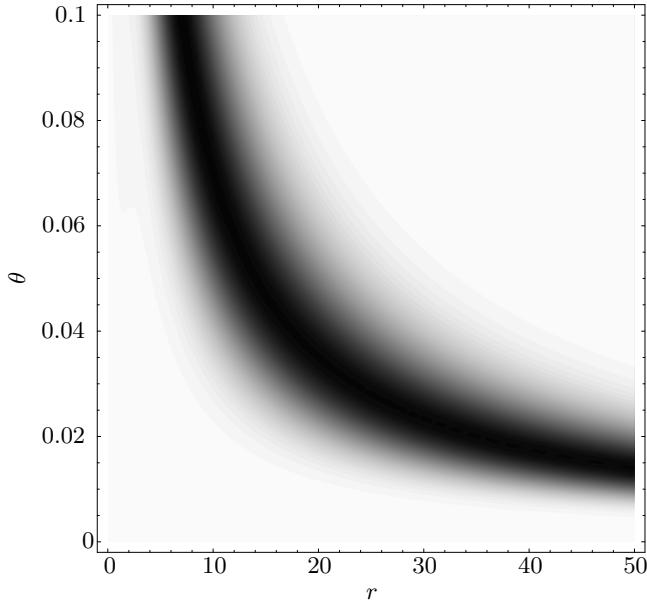


FIG. 3: The skewness is close to zero everywhere (darker equals more negative), except around a curve $r^2 \approx (2\theta^2)^{-1}$, where $\gamma_1 < 0$. In the non-phase modulated parity gate regime of operation $r \gtrsim \theta^{-2}$ the skewness is negligible.

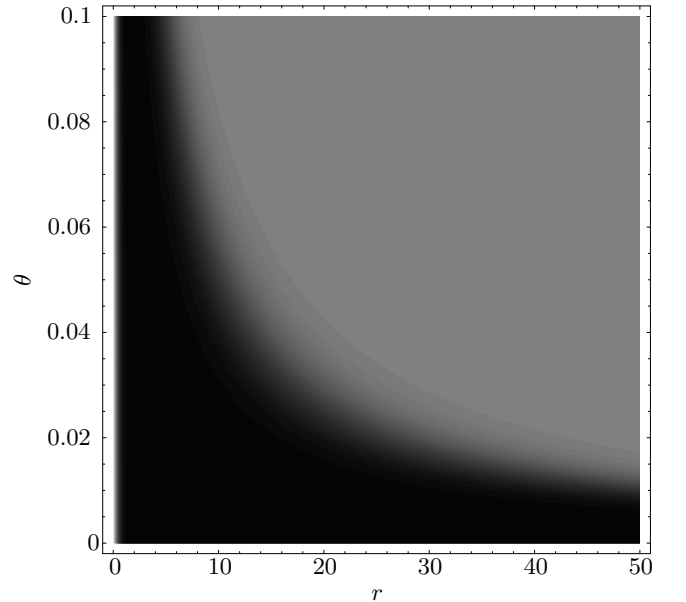


FIG. 4: The kurtosis in the parameter regime $r \gtrsim (2\theta)^{-2}$ is approximately -1.5 (darker is more negative), which means that the distribution is platykurtic. The values of the probability distribution tend to lie further away from the mean.

$\exp(-\zeta)$ and the resolution criterium becomes

$$|\langle \hat{x}'_\lambda \rangle_e - \langle \hat{x}'_\lambda \rangle_o| e^\zeta > (\Delta \hat{x}'_\lambda)_o + \max(\Delta \hat{x}'_\lambda)_e. \quad (\text{V.1})$$

The down-side of using squeezing is that if ζ is fairly large and λ is not chosen sufficiently accurate (which, we have seen, is rather difficult), it may exacerbate the indistinguishability of the peaks. Squeezing in the conjugate quadrature will result in an enhancement of the variance with a factor $\exp(\zeta)$. In a rotated frame $\hat{x}_\varphi = \cos \varphi \hat{x} + \sin \varphi \hat{x}_{\pi/2}$ the variance of a squeezed coherent state becomes

$$(\Delta \hat{x}_\varphi)^2 = \cos^2 \varphi e^{-2\zeta} + \sin^2 \varphi e^{2\zeta}. \quad (\text{V.2})$$

If the use of squeezing is to outperform the setup without squeezing, the offset in the rotation angle must be smaller than $\tan \varphi = \exp(-\zeta)$.

VI. CONCLUSIONS

Typically, weak nonlinear gates that do not propose to use EIT materials involve a self-phase modulation of the bus mode. In typical nonlinear materials this effect is half the size of the cross-phase modulation. Also, the cross-phase modulation (and hence the self-phase modulation)

may not be easily switched in the course of operating the gate. This means that the gate must be redesigned to operate with two successive identical phase shifts. In this case the self-phase modulation no longer cancels, as it cannot be switched in sign independently from the cross-phase modulation.

In this paper, it was shown that a typical self-phase modulation half the size of the cross-phase modulation will remove the distinguishability of the two parity measurement outcomes. This implementation of the gate will therefore not work without extra quadrature squeezing, which in turn presents difficulties regarding the tuning of the quadrature. The Wigner function and the marginal quadrature probability distribution for the bus state was constructed in terms of easily calculable generating functions.

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